ANL-7619

ANL-Idaho

SOSR

Rec.

File

ANL-7619

Argonne National Laboratory

VIBRATION OF A BEAM WITH MOTION-CONSTRAINT STOPS

by

S. S. Chen, G. S. Rosenberg,
and M. W. Wambsganss

The facilities of Argonne National Laboratory are owned by the United States Government. Under the terms of a contract (W-31-109-Eng-38) between the U. S. Atomic Energy Commission, Argonne Universities Association and The University of Chicago, the University employs the staff and operates the Laboratory in accordance with policies and programs formulated, approved and reviewed by the Association.

Printed in the United States of America
Available from
National Technical Information Service
U.S. Department of Commerce
Springfield, Virginia 22151
Price: Printed Copy $3.00; Microfiche $0.65

ANL-7619
Reactor Technology

ARGONNE NATIONAL LABORATORY
9700 South Cass Avenue
Argonne, Illinois 60439

VIBRATION OF A BEAM WITH MOTION-CONSTRAINT STOPS

by

S. S. Chen, G. S. Rosenberg,
and M. W. Wambsganss

Engineering and Technology Division

March 1970

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

NOMENCLATURE

Symbol	Description
a	Distance from left support to stop
b	Distance from right support to stop
d_i	Inside diameter of tube
d_o	Outside diameter of tube
e, e_1, e_2, e_i	Distance between stop and beam
$f(x), f(\xi)$	Distribution of external force; normalized to give $\int_0^{\ell} f(x)\,dx = \ell$
$k_i, \bar{k}_i, \bar{k}_n$	Eigenvalues for beam vibration
ℓ	Length of beam
m	Mass per unit length of beam
$q, q_n, \bar{q}_n, \tilde{q}_n$	Normal coordinates
r	Dimensionless distance
t	Time
u	Translational velocity of moving beam
v_0	Dimensionless initial velocity
v_1	Dimensionless velocity at contact with stop
$w, \bar{w}$	Dimensionless displacement
w_0	Dimensionless initial displacement
w_1	Displacement at instant beam strikes the stop
x	Axial coordinate
y	Transverse displacement
y_0	Initial displacement
$\dot{y}_0$	Initial velocity
z	Distance from neutral axis to fiber
$A(\tau)$	Dimensionless amplitude
C	Modulus of external damping
C_1, C_2, C_3	Constants computed by Eqs. 30 and 35
C_k	Lift coefficient
C_{ni}	Constants computed by Eqs. 72
C_σ	Bendi stress coefficient
C_V	Shearing force coefficient
C_I	Impact coefficient
$D_c(A), D_s(A), E_c(A), E_s(A), F_c(A), F_s(A)$	Constants computed by Eqs. 46
E	Young's modulus of elasticity
EI	Flexural rigidity
$F(x,t)$	External force per unit length
F_0	Mean amplitude of distributed external force

Symbol	Description
I	Moment of inertia
I_0	Impact
K_b	Stiffness constant for beam
K_s	Spring stiffness constant
M	"Added" mass of fluid
$M_n, \bar{M}_n$	Constants computed by Eqs. 16 and 26
N_s	Strouhal number
$Q, \bar{Q}$	Dimensionless forcing function
R	Mean amplitude of dimensionless forcing function $= \frac{F_0 \ell^3}{EI}$
U	Crossflow velocity
U_0	Mean crossflow velocity
V	Shearing force
α, β	Dimensionless damping coefficients
$\epsilon, \bar{\epsilon}, \bar{\epsilon}_i$	Dimensionless distance between beam and stops
ζ	Perturbation variable
$\zeta_n, \bar{\zeta}_n$	Modal damping factor
μ	Perturbation variable
λ	Modulus of internal damping
ξ	Dimensionless axial coordinate
ρ	Fluid density
σ	Bending stress
τ	Dimensionless time
τ_0	Dimensionless time when the beam passes the natural equilibrium position
τ_s	Dimensionless time when the beam strikes the stop
τ_e	Dimensionless time when the beam reaches maximum deflection
τ_D	Dimensionless time when the beam departs from stop
ω	Forcing frequency
$\varphi_n, \bar{\varphi}_n$	Normal modes
$\psi(\tau)$	Phase angle
Ω	Dimensionless forcing frequency
$\Omega_n, \bar{\Omega}_n$	Dimensionless natural frequencies of the beam

Eigenvalues for beam

Length of beam

Mass per unit length

Natural frequency

Time

with slab

VIBRATION OF A BEAM WITH MOTION-CONSTRAINT STOPS

by

S. S. Chen, G. S. Rosenberg, and M. W. Wambsganss

ABSTRACT

As the result of clearances imposed by manufacturing practices and design considerations, reactor fuel pins are susceptible to impacting with their supporting grid members, and heat-exchanger tubes can be expected to contact flow-directing baffles, which also serve as their intermediate supports. This intermittent contact will affect the vibrational characteristics and response of the component. To gain insight into the dynamics of the phenomenon, a theoretical analysis of a sinusoidally force-excited beam with motion-limiting stops is performed and a method of solution is presented. The "gross" motion is obtained through reduction of the system to a bilinear vibrating model. Information obtained from this model is used in a classical modal analysis in which many modes are included to give the dynamic stresses and impact in explicit form. Numerical results are given, and the character of impact is also investigated. The method is applied to study the tube-baffle interaction induced by crossflow in the EBR-II superheater.

I. INTRODUCTION

Various internal components of reactors and associated plant equipment, such as fuel, control rods, and heat-exchanger tubes, are long, slender members that derive their lateral support from intermittent, grid-type spacers or, with heat-exchanger tubes, baffle plates. Manufacturing and fabrication practices require a minimum clearance between component parts. Therefore, vibrations make the components susceptible to impacting and rubbing with the support, which could cause them to fail structurally by wear and fretting. Such impacting affects a component's vibrational characteristics thus further complicating analysis. To gain insight into the dynamic response of reactor system components that are subject to impacting with support members, a theoretical analysis of a simply-supported beam with motion-limiting stops was undertaken. The objectives of the study were to determine the dynamic stresses in the beam and impact at the stop and to investigate the effects of various parameters on the response of the beam.

Numerous attempts have been made to determine the impact and the stresses produced when a mass strikes a beam;[1-3] many of the results are described in a book on impact by Goldsmith.[4] However, an important inverse problem involving a vibrating beam striking a stop was untouched until some 20 years ago when serious dynamic-landing load problems were encountered in large landplanes and seaplanes. Useful methods of analysis[5,6] are available for the designer in determining transient landing stresses in airplanes, but the methods are based on the assumption that the landing impact force is independent of the airplane structural flexibility. This avoids inherent complexities such as those resulting from coupling of the airplane structure and the nonlinear properties of the landing gear of the landplane. More recently, Deas[7] studied the stresses in a beam during impact with a stop. He obtained the bending stress for impact of the free end of a free-pinned beam on a stop. Nevertheless, there is still a lack of published information regarding the impact and the dynamic response of a beam vibrating between two deflection resistors or stops.

Vibration of a beam with motion-limiting stops is complicated by the fact that the vibration is nonlinear in nature. The nonlinearity is due to the constraint imposed on the motion and indentation at the stop. If the local deformation at the contact area, as described by the Hertz law,[8] is incorporated in the theory, it will be necessary to solve a nonlinear integral equation or to analyze the response of a beam on a nonlinear spring.[9] To gain insight, several approximations are made. The local deformation is neglected, and a rigid stop is assumed; from the solution of a flying beam striking a stop, we find that this assumption is justified. Then a one-mode approximation is used to investigate the steady-state response of the beam. After the one-mode response is obtained, more modes are included in the analysis to more accurately determine dynamic stresses and impact.

II. MATHEMATICAL FORMULATION AND METHOD OF SOLUTION

The equation of motion for a vibrating beam of uniform cross section, including the effects of damping and distributed load is assumed to be

$$EI\,\frac{\partial^4 y}{\partial x^4} + \lambda I\,\frac{\partial^5 y}{\partial x^4\,\partial t} + C\,\frac{\partial y}{\partial t} + m\,\frac{\partial^2 y}{\partial t^2} = F(x,t). \tag{1}$$

Rotatory inertia and shear deformation are neglected. The theory of vibration and of the associated impact can be developed in a general manner for various conditions of support. Here, a simply-supported beam, as illustrated in Fig. 1, is taken as a vehicle to demonstrate the method of analysis.

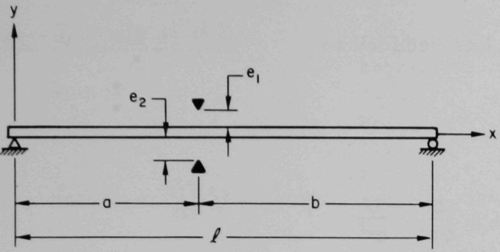

Fig. 1

Simply-supported Beam with Motion-constraint Stops. ANL Neg. No. 113-3286.

The boundary conditions for a simply-supported beam are

$$\left.\begin{array}{ll} y(0,t) = 0; & y(\ell,t) = 0; \\ \dfrac{\partial^2 y}{\partial x^2}(0,t) = 0; & \dfrac{\partial^2 y}{\partial x^2}(\ell,t) = 0. \end{array}\right\} \tag{2}$$

Because the beam is constrained by the stops at $x = a$, an additional condition is required at that section; neglecting local deformation at the contact point, we can write

$$-e_2 \le y(a,t) \le e_1. \tag{3}$$

Assume that the initial conditions are

$$\left.\begin{array}{l} y(x,0) = y_0(x) \\ \text{and} \\ \dfrac{\partial y}{\partial t}(x,0) = \dot{y}_0(x) \end{array}\right\}. \tag{4}$$

Before we proceed with the analysis, it is desirable to express the terms in dimensionless form; accordingly we put

$$\left.\begin{array}{l} \xi = x/\ell, \quad \tau = t\left(\dfrac{EI}{m\ell^4}\right)^{1/2}, \quad Q = \dfrac{F\ell^3}{EI}, \\ w = y/\ell, \quad \alpha = \lambda\left(\dfrac{I}{Em\ell^4}\right)^{1/2}, \quad w_0 = y_0/\ell, \\ r = a/\ell, \quad \beta = C\left(\dfrac{\ell^4}{EIm}\right)^{1/2}, \quad v_0 = \left(\dfrac{m}{EI}\right)^{1/2}\dot{y}_0\ell, \\ \overline{\epsilon}_i = e_i/\ell, \text{ and } \Omega = \left(\dfrac{m}{EI}\right)^{1/2}\omega\ell^2. \end{array}\right\} \tag{5}$$

In dimensionless form, Eq. 1 is reduced to

$$Lw = Q, \tag{6}$$

where

$$L = \frac{\partial^4}{\partial \xi^4} + \alpha \frac{\partial^5}{\partial \xi^4 \partial \tau} + \beta \frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial \tau^2}.$$

The operator L is defined in the domain

$$D(L): \; w(\xi, \tau), \quad \left\{ \begin{array}{l} 0 \le \tau \\ 0 \le \xi \le 1 \end{array} \right. \tag{7}$$

with the following boundary and initial conditions:

$$\left. \begin{array}{l} w(0, \tau) = \dfrac{\partial^2 w}{\partial \xi^2}(0, \tau) = 0, \\ \\ w(1, \tau) = \dfrac{\partial^2 w}{\partial \xi^2}(1, \tau) = 0, \end{array} \right\} \tag{8}$$

$$-\bar{\varepsilon}_2 \le w(r, \tau) \le \bar{\varepsilon}_1, \tag{9}$$

$$\left. \begin{array}{l} w(\xi, 0) = w_0(\xi), \\ \text{and} \\ \dfrac{\partial w}{\partial \tau}(\xi, 0) = v_0(\xi). \end{array} \right\} \tag{10}$$

The problem is to find $w(\xi, \tau)$ which satisfies Eqs. 6 and 8-10 such that Lw is in the domain of L.

With no loss of generality, it is assumed that

$$\left. \begin{array}{l} -\bar{\varepsilon}_2 < w(r, 0) < \bar{\varepsilon}_1 \\ \text{and} \\ v_0(r) > 0 \end{array} \right\}. \tag{11}$$

The problem can be described in two regimes. If the beam does not strike the stop, i.e., $-\bar{\varepsilon}_2 < w(r, \tau) < \bar{\varepsilon}_1$, Eq. 9 can be ignored. After the beam strikes the stop, i.e., $w(r, \tau) = \bar{\varepsilon}_1$, or $w(r, \tau) = -\bar{\varepsilon}_2$, the beam vibrates as a two-span continuous beam. These two regimes are described as follows:

1. <u>For $-\bar{\epsilon}_2 < w(r, \tau) < \bar{\epsilon}_1$</u>

$$Lw = Q \tag{12}$$

and

$$\left.\begin{aligned}
w(0,\tau) &= \frac{\partial^2 w}{\partial \xi^2}(0,\tau) = 0, \\
w(1,\tau) &= \frac{\partial^2 w}{\partial \xi^2}(1,\tau) = 0, \\
w(\xi,0) &= w_0(\xi), \\
&\text{and} \\
\frac{\partial w}{\partial \tau}(\xi,0) &= v_0(\xi).
\end{aligned}\right\} \tag{13}$$

The undamped system constitutes a self-adjoint problem. Several methods are available for analytical solution, such as the normal-mode method, Green's function, or finite transform. The solution is summarized as follows:

$$\left.\begin{aligned}
&\text{Natural frequencies} && \Omega_n = n^2\pi^2 \\
&\text{Displacement} && w(\xi,\tau) = \sum_n q_n(\tau)\varphi_n(\xi) \\
& && n = 1, 2, 3, 4 \ldots \\
&\text{Normal modes} && \varphi_n(\xi) = \sin n\pi\xi
\end{aligned}\right\} \tag{14}$$

where q_n is the normal coordinate, which is the solution of the following equations:

$$\left.\begin{aligned}
\frac{\partial^2 q_n}{\partial \tau^2} + 2\zeta_n\Omega_n \frac{\partial q_n}{\partial \tau} + \Omega_n^2 q_n &= \frac{1}{M_n}\int_0^1 Q(\xi,\tau)\varphi_n(\xi)\, d\xi, \\
q_n(0) &= \frac{1}{M_n}\int_0^1 w_0(\xi)\varphi_n(\xi)\, d\xi, \\
&\text{and} \\
\frac{\partial q_n(0)}{\partial \tau} &= \frac{1}{M_n}\int_0^1 v_0(\xi)\varphi_n(\xi)\, d\xi,
\end{aligned}\right\} \tag{15}$$

where

$$\left.\begin{aligned} \zeta_n &= \frac{\alpha\Omega_n}{2} + \frac{\beta}{2\Omega_n} \\ &\text{and} \\ M_n &= \int_0^1 \varphi_n^2(\xi)\, d\xi \end{aligned}\right\}. \tag{16}$$

2. <u>For $w(r,\tau) = -\bar{\epsilon}_2$, or $w(r,\tau) = \bar{\epsilon}_1$</u>

From the displacement equation in Eqs. 14, we can find the time when the beam strikes the stop; let it be designated as τ_s. As soon as the beam strikes the stop, an additional condition at the stop has to be imposed on the system. The displacement and velocity at τ_s are taken as the initial conditions of the second regime. Therefore, we can write

$$Lw(\xi,\tau) = Q(\xi,\tau) \tag{17}$$

and

$$\left.\begin{aligned} w(0,\tau) &= \frac{\partial^2 w}{\partial \xi^2}(0,\tau) = 0, \\ w(1,\tau) &= \frac{\partial^2 w}{\partial \xi^2}(1,\tau) = 0, \\ w(r,\tau) &= \bar{\epsilon}_1 \text{ or } -\bar{\epsilon}_2, \\ w(\xi,\tau_s) &= w_1(\xi), \\ &\text{and} \\ \frac{\partial w}{\partial \tau}(\xi,\tau)\Big|_{\tau=\tau_s} &= v_1(\xi), \end{aligned}\right\} \tag{18}$$

where

$$\left.\begin{aligned} w_1(\xi) &= \sum_n q_n(\tau_s)\varphi_n(\xi) \\ &\text{and} \\ v_1(\xi) &= \sum_n \frac{\partial q_n}{\partial \tau}(\tau_s)\varphi_n(\xi) \end{aligned}\right\}. \tag{19}$$

Equations 17 and 18 describe a linear system with nonhomogeneous boundary conditions. To eliminate these conditions, let

$$w(\xi,\tau) = \overline{w}(\xi,\tau) + w_1(\xi). \tag{20}$$

If we substitute Eq. 20 into Eqs. 17-19 and use the properties of $w_1(\xi)$, the system description becomes

$$L\overline{w}(\xi,\tau) = \overline{Q}(\xi,\tau) \tag{21}$$

and

$$\left.\begin{aligned} \overline{w}(0,\tau) &= \frac{\partial^2\overline{w}}{\partial\xi^2}(0,\tau) = 0, \\ \overline{w}(1,\tau) &= \frac{\partial^2\overline{w}}{\partial\xi^2}(1,\tau) = 0, \\ \overline{w}(r,\tau) &= 0, \\ \overline{w}(\xi,\tau_s) &= 0, \\ &\text{and} \\ \frac{\partial\overline{w}}{\partial\tau}(\xi,\tau)\bigg|_{\tau=\tau_s} &= v_1(\xi), \end{aligned}\right\} \tag{22}$$

where

$$\overline{Q}(\xi,\tau) = Q(\xi,\tau) - \sum_n \Omega_n^2 q_n(\tau_s)\varphi_n(\xi). \tag{22.a}$$

Again we obtain a self-adjoint system for the undamped problem. The normal-mode method of solution is suitable for solving this system. On representing $\overline{w}(\xi,\tau)$ by

$$\overline{w}(\xi,\tau) = \sum_n \overline{q}_n(\tau)\varphi_n(\xi), \tag{23}$$

we obtain the following results:

$$\left.\begin{aligned} &\text{Natural frequencies: } \overline{\Omega}_n = \overline{k}_n^2 \\ &\text{Displacement: } w(\xi,\tau) = \sum_n \overline{q}_n(\tau)\overline{\varphi}_n(\xi) + w_1(\xi) \\ &\qquad n = 1, 2, 3, \ldots \end{aligned}\right\} \tag{24}$$

(Contd.)

(Contd.)

$$\text{Normal modes:}\quad \bar{\varphi}_n = \sin(\bar{k}_n\xi) - \frac{\sin(\bar{k}_n r)}{\sinh(\bar{k}_n r)}\sinh(\bar{k}_n\xi), \qquad 0 \le \xi \le r$$

$$\bar{\varphi}_n = \sin(\bar{k}_n r)\{-\cot[\bar{k}_n(1-r)]\sin[\bar{k}_n(\xi - r)] - \cos[\bar{k}_n(\xi - r)] + \cosh[\bar{k}_n(\xi - r)] + \coth[\bar{k}_n(1-r)]\sinh[\bar{k}_n(\xi - r)]\}, \qquad r \le \xi \le 1, \tag{24}$$

where $\bar{k}_n$ is the solution of

$$\cot(\bar{k}_n r) - \coth(\bar{k}_n r) = -\cot[\bar{k}_n(1-r)] + \coth[\bar{k}_n(1-r)] \tag{25}$$

and

$$\frac{\partial^2 \bar{q}_n}{\partial \tau^2} + 2\bar{\zeta}_n\bar{\Omega}_n\frac{\partial \bar{q}_n}{\partial \tau} + \bar{\Omega}_n^2\bar{q}_n = \frac{1}{\bar{M}_n}\int_0^1 \bar{Q}(\xi,\tau)\bar{\varphi}_n(\xi)\,d\xi,$$

$$\bar{q}_n(\tau_s) = 0,$$

$$\frac{\partial \bar{q}_n}{\partial \tau}(\tau_s) = \frac{1}{\bar{M}_n}\int_0^1 v_1(\xi)\bar{\varphi}_n(\xi)\,d\xi, \tag{26}$$

$$\bar{M}_n = \int_0^1 \bar{\varphi}_n^2(\xi)\,d\xi,$$

and

$$\bar{\zeta}_n = \frac{\alpha\bar{\Omega}_n}{2} + \frac{\beta}{2\bar{\Omega}_n}.$$

From the displacement equation in Eqs. 24, we can find the time τ_D (measured from the time of contact) when the beam departs from the stop. At this time, it reverts to regime 1. Hence we are able to find the response of the beam at any time by using step-by-step analysis. However, if a large number of modes are included, it is very tedious to study the steady-state response by this method. Therefore, the following procedure is followed to avoid complexities:

1. From a "first-mode approximation" (i.e., the beam is assumed to respond as a system with a single degree of freedom), the "gross" response of the beam is obtained.

2. With the information obtained from first-mode approximation, more modes are included to find the dynamic stresses and impact.

III. ONE-MODE APPROXIMATION

A. Reduction to a Bilinear Vibrating Model

A simply-supported beam with a symmetric stop at midspan is considered, i.e., $r = \frac{1}{2}$, $\bar{\epsilon}_1 = \bar{\epsilon}_2 = \bar{\epsilon}$. If only the first mode is taken, we obtain from Eqs. 14-16, 20, 23, 24, and 26 the following results (see Fig. 2):

1. For $-\bar{\epsilon} < w(\frac{1}{2},\tau) < \bar{\epsilon}$, $\quad 0 \le \tau \le \tau_s$

$$\left.\begin{aligned}
w(\xi,\tau) &= q_1(\tau)\varphi_1(\xi), \\
\dot{w}(\xi,\tau) &= \dot{q}_1(\tau)\varphi_1(\xi), \\
\ddot{q}_1 + 2\zeta_1\Omega_1\dot{q}_1 + \Omega_1^2 q_1 &= \frac{1}{M_1}\int_0^1 Q(\xi,\tau)\varphi_1\, d\xi, \\
q_1(0) &= \frac{1}{M_1}\int_0^1 w_0(\xi)\varphi_1\, d\xi,
\end{aligned}\right\}$$

and

$$\dot{q}_1(0) = \frac{1}{M_1}\int_0^1 v_0(\xi)\varphi_1\, d\xi. \tag{27}$$

2. For $w(\frac{1}{2},\tau) = \pm\bar{\epsilon}$, $\tau_s \le \tau \le \tau_s + \tau_D$

$$\left.\begin{aligned}
w(\xi,\tau) &= \bar{q}_1(\tau)\bar{\varphi}_1(\xi) + q_1(\tau_s)\varphi_1(\xi), \\
\dot{w}(\xi,\tau) &= \dot{\bar{q}}_1(\tau)\bar{\varphi}_1(\xi), \\
\ddot{\bar{q}}_1 + 2\bar{\zeta}_1\bar{\Omega}_1\dot{\bar{q}}_1 + \bar{\Omega}_1^2\bar{q}_1 &= \frac{1}{\bar{M}_1}\int_0^1 \bar{Q}(\xi,\tau)\bar{\varphi}_1\, d\xi, \\
\bar{q}_1(\tau_s) &= 0, \\
\dot{\bar{q}}_1(\tau_s) &= \frac{\dot{q}(\tau_s)}{\bar{M}_1}\int_0^1 \varphi_1\bar{\varphi}_1\, d\xi,
\end{aligned}\right\}$$

and

$$\bar{Q}(\xi,\tau) = Q(\xi,\tau) - q_1(\tau_s)\Omega_1^2\varphi_1(\xi). \tag{28}$$

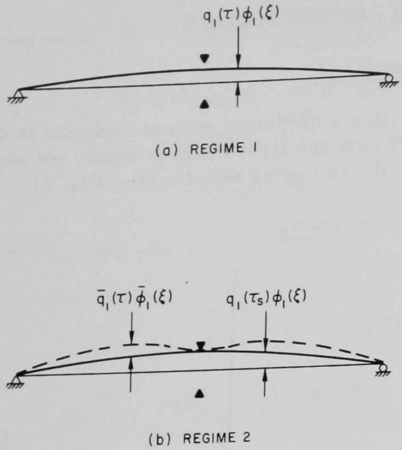

Fig. 2
Displacement of Beam Using One-mode Approximation. ANL Neg. No. 113-3282.

To incorporate Eqs. 27 and 28 into one system, it is necessary to have continuous conditions at $\tau = \tau_s$ such that the normal coordinates q_1 and $\bar{q}_1$ for the two regimes can be combined into one. To accomplish this, we define a new normal coordinate, $\tilde{q}_1(\tau)$, such that

$$\left.\begin{aligned} \tilde{q}_1(\tau_s) &= q_1(\tau_s) \\ &\text{and} \\ \dot{\tilde{q}}_1(\tau_s) &= \dot{q}_1(\tau_s) \end{aligned}\right\}. \tag{29}$$

These conditions are satisfied by taking

$$\left.\begin{aligned} &\tilde{q}_1(\tau) = \frac{1}{C_1}\bar{q}_1(\tau) + q_1(\tau_s), \\ &\text{where} \\ &C_1 = \frac{\int_0^1 \bar{\varphi}_1\varphi_1\, d\xi}{\int_0^1 \bar{\varphi}_1^2\, d\xi}; \\ &\text{also let} \\ &\tilde{\varphi}_1(\xi) = C_1\bar{\varphi}_1(\xi). \end{aligned}\right\} \tag{30}$$

Substituting Eqs. 30 into Eqs. 28, we obtain as the new system of equations for $w(\frac{1}{2}, \tau) = \pm\epsilon$, $\tau_s \leq \tau \leq \tau_s + \tau_D$,

$$\left.\begin{aligned} w(\xi,\tau) &= [\tilde{q}_1(\tau) - q_1(\tau_s)]\,\tilde{\varphi}_1(\xi) + q_1(\tau_s)\varphi_1(\xi), \\ \mathring{w}(\xi,\tau) &= \mathring{\tilde{q}}_1(\tau)\tilde{\varphi}_1(\xi), \\ \mathring{\mathring{\tilde{q}}}_1 + 2\bar{\zeta}_1\bar{\Omega}_1\mathring{\tilde{q}}_1 + \bar{\Omega}_1^2\left[\tilde{q}_1 - q_1(\tau_s)\left(1 - \frac{\Omega_1^2}{\bar{\Omega}_1^2}\right)\right] &= \frac{1}{C_1\bar{M}_1}\int_0^1 Q\bar{\varphi}_1\,d\xi, \\ \tilde{q}_1(\tau_s) &= q_1(\tau_s), \\ \text{and} \qquad \mathring{\tilde{q}}_1(\tau_s) &= \dot{q}_1(\tau_s). \end{aligned}\right\} \qquad (31)$$

Assume the forcing function is periodic and of the form

$$Q(\xi,\tau) = Rf(\xi)\cos\Omega\tau, \qquad (32)$$

where R is a constant and

$$\int_0^1 f(\xi)\,d\xi = 1.$$

Compare Eqs. 27 with Eqs. 31, and observe that the normal coordinates q_1 and $\tilde{q}_1$ are governed by the single equation

$$\ddot{q} + f_1(\dot{q}) + f_2(q) = f_3\cos\Omega\tau, \qquad (33)$$

where

$$\left.\begin{aligned} \text{for } |q| \leq \epsilon,\quad f_1(\dot{q}) &= 2\zeta_1\Omega_1\dot{q}, \\ f_2(q) &= \Omega_1^2 q, \\ \text{and} \qquad f_3(q) &= C_2R; \\ \text{for } |q| \geq \epsilon,\quad f_1(\dot{q}) &= 2\bar{\zeta}_1\bar{\Omega}_1\dot{q}, \\ f_2(q) &= \bar{\Omega}_1^2(q-\epsilon) + \Omega_1^2\epsilon, \\ \text{and} \qquad f_3(q) &= C_3R; \end{aligned}\right\} \qquad (34)$$

and

$$\varepsilon = \bar{\varepsilon}/\varphi_1(\tfrac{1}{2}),$$

$$\left.\begin{aligned} C_2 &= \frac{\int_0^1 f(\xi)\varphi_1(\xi)\,d\xi}{\int_0^1 \varphi_1^2(\xi)\,d\xi}, \\ &\text{and} \\ C_3 &= \frac{\int_0^1 f(\xi)\bar{\varphi}_1(\xi)\,d\xi}{\int_0^1 \bar{\varphi}_1\varphi_1\,d\xi}. \end{aligned}\right\} \tag{35}$$

The displacement of the beam is characterized by $q(\tau)$

$$\left.\begin{aligned} w(\xi,\tau) &= q(\tau)\varphi_1(\xi), && \text{for } |q| \le \varepsilon; \\ w(\xi,\tau) &= C_1[q(\tau)-\varepsilon]\,\bar{\varphi}_1(\xi) + \varepsilon\varphi_1(\xi), && \text{for } |q| \ge \varepsilon. \end{aligned}\right\} \tag{36}$$

Equation 33 is the same as the equation of motion of a unit mass on a bilinear spring excited by a piecewise linear force as shown in Fig. 3.

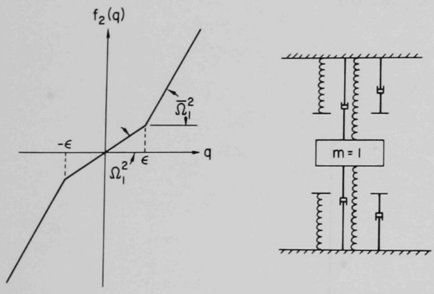

Fig. 3. Restoring Force and Its Equivalent System. ANL Neg. No. 113-3278.

B. Van der Pol and Krylov-Bogoliubov Methods

When $q < \varepsilon$, the system described by Eq. 33 is linear and the solution will be cosinusoidal in nature. For $q > \varepsilon$, it is reasonable to assume that the trigonometric character of response will be preserved. Therefore the Van der Pol and Krylov-Bogoliubov Methods[10] are applicable for obtaining the solution. Let the solution to Eqs. 33 and 34 be approximated by

$$q(\tau) = A(\tau)\cos\theta, \tag{37}$$

where

$$\theta = \Omega\tau - \psi(\tau) \tag{38}$$

and $A(\tau)$ and $\psi(\tau)$ are slowly-varying functions of τ. Then,

$$\dot{q} = \dot{A}(\tau) \cos \theta - \Omega A(\tau) \sin \theta + \dot{\psi}(\tau) A(\tau) \sin \theta. \tag{39}$$

Let us further assume, as a first approximation, that

$$\dot{A}(\tau) \cos \theta + \dot{\psi}(\tau) A(\tau) \sin \theta = 0; \tag{40}$$

then

$$\dot{q} = -\Omega A(\tau) \sin \theta \tag{41}$$

and

$$\ddot{q} = -\Omega \dot{A}(\tau) \sin \theta - \Omega^2 A(\tau) \cos \theta + \Omega A(\tau) \dot{\psi}(\tau) \cos \theta. \tag{42}$$

Substitution of Eq. 42 into Eq. 33 gives

$$-\Omega \dot{A} \sin \theta - \Omega^2 A \cos \theta + \Omega A \dot{\psi} \cos \theta + f_1(\dot{q}) + f_2(q) = f_3(q) \cos \Omega \tau. \tag{43}$$

Multiplying Eq. 40 by $\Omega \sin \theta$ and Eq. 43 by $\cos \theta$ and adding, and then averaging over one cycle of θ, we obtain

$$\Omega \dot{\psi} A - \frac{\Omega^2 A}{2} + \frac{1}{2\pi} \int_0^{2\pi} f_1(\dot{q}) \cos \theta \, d\theta + \frac{1}{2\pi} \int_0^{2\pi} f_2(q) \cos \theta \, d\theta$$
$$= \frac{1}{2\pi} \int_0^{2\pi} f_3(q) \cos \theta \cos (\theta + \psi) \, d\theta. \tag{44}$$

Similarly, multiplying Eq. 40 by $\Omega \cos \theta$ and Eq. 43 by $\sin \theta$ and adding, and then averaging over one cycle of θ, gives

$$-\Omega \dot{A} + \frac{1}{2\pi} \int_0^{2\pi} f_1(\dot{q}) \sin \theta \, d\theta + \frac{1}{2\pi} \int_0^{2\pi} f_2(q) \sin \theta \, d\theta$$
$$= \frac{1}{2\pi} \int_0^{2\pi} f_3(q) \sin \theta \cos (\theta + \psi) \, d\theta. \tag{45}$$

If we let

$$\left.\begin{aligned} D_c(A) &= \frac{1}{\epsilon\pi\Omega} \int_0^{2\pi} f_1(\dot{q}) \cos \theta \, d\theta, \\ D_s(A) &= \frac{1}{\epsilon\pi\Omega} \int_0^{2\pi} f_1(\dot{q}) \sin \theta \, d\theta, \end{aligned}\right\} \quad \begin{matrix}(46)\\ \text{(Contd.)}\end{matrix}$$

(Contd.)

$$
\left.
\begin{aligned}
E_c(A) &= \frac{1}{\pi\varepsilon}\int_0^{2\pi} f_2(q)\cos\theta\, d\theta, \\
E_s(A) &= \frac{1}{\pi\varepsilon}\int_0^{2\pi} f_2(q)\sin\theta\, d\theta, \\
F_c(A) &= \frac{1}{\pi\varepsilon\cos\psi}\int_0^{2\pi} f_3(q)\cos\theta\cos(\theta+\psi)\, d\theta,
\end{aligned}
\right\} \quad (46)
$$

and

$$
F_s(A) = \frac{1}{\pi\varepsilon\sin\psi}\int_0^{2\pi} f_3(q)\sin\theta\cos(\theta+\psi)\, d\theta,
$$

Eqs. 44 and 45 become

$$
2\Omega\dot{\psi}\frac{A}{\varepsilon} - \Omega^2\frac{A}{\varepsilon} + \Omega D_c(A) + E_c(A) = F_c(A)\cos\psi
$$

and

$$
-2\Omega\frac{\dot{A}}{\varepsilon} + \Omega D_s(A) + E_s(A) = F_s(A)\sin\psi \quad (47)
$$

Equations 47 may be regarded as a complete statement of the problem in terms of A and Ω. The analysis is valid for any type of system where $f_1(\dot{q})$, $f_2(q)$, and $f_3(q)$ may have some nonlinear forms. For the bilinear system described by Eqs. 33, the integrals of Eqs. 46 yield

$$
\begin{aligned}
D_c(A) &= 0, \\
D_s(A) &= -2\zeta_1\Omega_1\frac{A}{\varepsilon}, \\
E_c(A) &= \Omega_1^2\frac{A}{\varepsilon} && \text{for } A \le \varepsilon, \\
&= \frac{4}{\pi}\left[\left(\frac{\theta_1}{2} + \frac{\sin 2\theta_1}{4}\right)(\bar{\Omega}_1^2 - \Omega_1^2)\sec\theta_1 + \frac{\pi}{4}\Omega_1^2\sec\theta_1 \right. \\
&\quad \left. -(\bar{\Omega}_1^2 - \Omega_1^2)\sin\theta_1\right] && \text{for } A \ge \varepsilon, \\
E_s(A) &= 0,
\end{aligned}
\quad (48)
$$

(Contd.)

$$
\left.
\begin{aligned}
F_c(A) &= C_2\frac{R}{\epsilon} && \text{for } A \le \epsilon,\\
&= \frac{4}{\pi}\left[\frac{R}{\epsilon}(C_3 - C_2)\left(\frac{\theta_1}{2} + \frac{1}{4}\sin 2\theta_1\right) + \frac{\pi}{4}C_2\right] && \text{for } A \ge \epsilon,\\
F_s(A) &= -C_2\frac{R}{\epsilon} && \text{for } A \le \epsilon,\\
\text{and} &\\
&= -\frac{4}{\pi}\frac{R}{\epsilon}\left[(C_3 - C_2)\left(\frac{\theta_1}{2} - \frac{1}{4}\sin 2\theta_1\right) + \frac{\pi}{4}C_2\right] && \text{for } A \ge \epsilon,
\end{aligned}
\right\} \quad \text{(Contd.)} \tag{48}
$$

where $\theta_1 = \cos^{-1}(\epsilon/A)$. Hence, Eqs. 47 reduce to

$$
\left.
\begin{aligned}
2\Omega\dot{\psi}\frac{A}{\epsilon} - \Omega^2\frac{A}{\epsilon} + E_c(A) &= F_c(A)\cos\psi\\
\text{and}\qquad &\\
-2\Omega\frac{\dot{A}}{\epsilon} + \Omega D_s(A) &= F_s(A)\sin\psi
\end{aligned}
\right\}. \tag{49}
$$

The steady-state response is obtained from Eqs. 49 by setting $\dot{A}$ and $\dot{\psi}$ equal to zero. This gives

$$
\left.
\begin{aligned}
-\Omega^2\frac{A_0}{\epsilon} + E_c(A_0) &= F_c(A_0)\cos\psi_0\\
\text{and}\qquad &\\
\Omega D_s(A_0) &= F_s(A_0)\sin\psi_0
\end{aligned}
\right\}, \tag{50}
$$

where the subscript $_0$ denotes steady-state values. From these two equations, the following phase angle and amplitude-frequency relation are obtained:

$$
\left.
\begin{aligned}
&\tan\psi_0 = \frac{F_c(A_0)\Omega D_s(A_0)}{F_s(A_0)\left[E_c(A_0) - \Omega^2\frac{A_0}{\epsilon}\right]}\\
&\text{and}\\
&\Omega^4 - 2\left[E_c(A_0)\frac{\epsilon}{A_0} - \tfrac{1}{2}D_s^2(A_0)\left(\frac{F_c(A_0)}{F_s(A_0)}\right)^2\left(\frac{\epsilon}{A_0}\right)^2\right]\Omega^2\\
&\quad + \left[E_c^2(A_0) - F_c^2(A_0)\right]\left(\frac{\epsilon}{A_0}\right)^2 = 0.
\end{aligned}
\right\} \tag{51}
$$

If $A_0 < \epsilon$, Eqs. 51 reduce to the familiar form

$$\left.\begin{aligned} \tan\psi_0 &= \frac{2\zeta_1\Omega_1\Omega}{\Omega_1^2 - \Omega^2} \\ &\text{and} \\ (\Omega^2 - \Omega_1^2)^2 + 4\zeta_1^2\Omega_1^2\Omega^2 &= \left(\frac{C_2 R}{A_0}\right)^2 \end{aligned}\right\}, \tag{51.a}$$

which is the steady-state reponse of a spring mass system.

C. Stability Analysis

Let

$$\left.\begin{aligned} A &= A_0 + \zeta \\ &\text{and} \\ \psi &= \psi_0 + \eta \end{aligned}\right\}, \tag{52}$$

where ζ and η are small perturbations on the steady-state parameters. Substituting Eqs. 52 into Eqs. 49 and neglecting higher-order terms, we obtain

$$\left.\begin{aligned} &2\Omega\frac{A_0}{\epsilon}\dot{\eta} + F_C(A_0)(\sin\psi_0)\eta + \left[E_C'(A_0) - \frac{\Omega^2}{\epsilon} - F_C'(A_0)\cos\psi_0\right]\zeta = 0 \\ &\text{and} \\ &2\Omega\frac{1}{\epsilon}\dot{\zeta} + [F_S'(A_0)\sin\psi_0 - \Omega D_S'(A_0)]\,\zeta - F_S(\cos\psi_0)\eta = 0 \end{aligned}\right\} \tag{53}$$

where

$$\left.\begin{aligned} E_C'(A_0) &= \left.\frac{\partial E_C(A)}{\partial A}\right|_{A = A_0}, \\ F_C'(A_0) &= \left.\frac{\partial F_C(A)}{\partial A}\right|_{A = A_0}, \\ F_S'(A_0) &= \left.\frac{\partial F_S(A)}{\partial A}\right|_{A = A_0}, \\ &\text{and} \\ D_S'(A_0) &= \left.\frac{\partial D_S(A)}{\partial A}\right|_{A = A_0}. \end{aligned}\right\} \tag{54}$$

To examine the behavior of the small perturbations ζ and η, let

$$\left.\begin{aligned} \zeta &= \bar{\zeta}e^{\lambda\tau} \\ &\text{and} \\ \eta &= \bar{\eta}e^{\lambda\tau} \end{aligned}\right\}. \tag{55}$$

Substitution of Eqs. 55 into Eqs. 53 yields

$$\left.\begin{aligned} &\left(2\Omega\lambda\frac{A_0}{\varepsilon} + F_c \sin\psi_0\right)\bar{\eta} + \left(E_c' - \frac{\Omega^2}{\varepsilon} - F_c' \cos\psi_0\right)\bar{\zeta} = 0 \\ &(-F_s \cos\psi_0)\bar{\eta} + \left(2\Omega\lambda\frac{1}{\varepsilon} + F_s' \sin\psi_0 - \Omega D_s'\right)\bar{\zeta} = 0 \end{aligned}\right\}. \tag{56}$$

The frequency equation is obtained by setting the determinant of the coefficients of $\bar{\eta}$ and $\bar{\zeta}$ equal to zero; this yields

$$(2\Omega\lambda)^2 + (2\Omega\lambda)G + H = 0, \tag{57}$$

where

$$\left.\begin{aligned} G &= 2\zeta_1\Omega_1\Omega\left(1 - \frac{F_c}{F_s}\right) + \frac{F_s'}{F_s}\varepsilon\Omega D_s \\ &\text{and} \\ H &= F_c \sin\psi_0(\varepsilon F_s' \sin\psi_0 - \Omega\varepsilon D_s') + F_s \cos\psi_0(\varepsilon E_c' - \Omega^2 - \varepsilon F_c' \cos\psi_0) \end{aligned}\right\}. \tag{58}$$

For a given set of Ω and A_0 obtained from Eq. 51, one can determine the value of λ from the characteristic equation, Eq. 57. It may be shown that

$$G > 0 \text{ for all } A_0. \tag{59}$$

If $H = 0$, one root of Eq. 57 is $\lambda_1 = 0$, and the other is negative real; the system is on the margin of the stable region. If $H < 0$, the eigenvalues λ_1 and λ_2 are both real and $\lambda_1\lambda_2 < 0$; the system will be unstable. If $H > 0$, the eigenvalues may be either real or complex-conjugate, but the real part of both roots will be negative; therefore, the system is stable.

IV. IMPACT AND DYNAMIC STRESSES OF THE BEAM

From Eqs. 36, 37, and 41, we have for the steady-state response of the one-mode approximation

$$\left.\begin{aligned} w(\xi,\tau) &= q(\tau)\varphi_1(\xi) && \text{for } |q| \le \epsilon, \\ &= C_1(q-\epsilon)\bar{\varphi}_1 + \epsilon\varphi_1 && \text{for } |q| \ge \epsilon, \\ q(\tau) &= A_0 \cos(\Omega\tau - \psi_0), \\ \text{and} \\ \dot{q}(\tau) &= \Omega A_0 \sin(\Omega\tau - \psi_0). \end{aligned}\right\} \tag{60}$$

To study the stress and shear force responses of the beam, additional modes must be included in the analysis to obtain sufficiently accurate results. The response will be characterized during a quarter of the period of the forcing frequency. Let τ_0, τ_s, and τ_e be defined as the times when the beam passes the natural equilibrium position, strikes the stop, and reaches the extreme position, respectively; thus

$$\left.\begin{aligned} q(\tau_0) &= 0, \\ q(\tau_s) &= \epsilon, \\ \text{and} \\ \dot{q}(\tau_e) &= 0, \end{aligned}\right\} \tag{61}$$

and

$$\left.\begin{aligned} \tau_0 &= \frac{1}{\Omega}\frac{3}{2}\pi + \psi_0, \\ \tau_s &= \frac{1}{\Omega}\left(2\pi + \psi_0 - \cos^{-1}\frac{\epsilon}{A_0}\right), \\ \text{and} \\ \tau_e &= \frac{1}{\Omega}(2\pi + \psi_0). \end{aligned}\right\} \tag{62}$$

1. $\underline{\tau_0 \le \tau \le \tau_s}$ (Without striking the stop)

The information, at $\tau = \tau_0$, of one-mode solution is taken as the initial condition of this regime. In Eqs. 60 and 32, let $\tau' = \tau - \tau_0$; we then have

$$
\left.\begin{aligned}
w(\xi,\tau')\Big|_{\tau'=0} &= q(\tau_0)\varphi_1(\xi), \\
\dot{w}(\xi,\tau')\Big|_{\tau'=0} &= \dot{q}(\tau_0)\varphi_1(\xi),
\end{aligned}\right\} \tag{63}
$$

and

$$
Q(\xi,\tau) = Rf(\xi)\cos\Omega\tau = Rf(\xi)\cos[\Omega(\tau'+\tau_0)] = Q(\xi,\tau'). \tag{64}
$$

Because the beam does not strike the stop, the system is described by Eqs. 12-16. In terms of τ', using Eqs. 63 as the initial conditions and Eqs. 64 as the forcing function, one obtains the solution

$$
w(\xi,\tau') = \sum_i q_i(\tau')\varphi_i(\xi), \qquad 0 \le \tau' \le \tau_s - \tau_0, \tag{65}
$$

where

$$
\left.\begin{aligned}
q_i(\tau') &= \left[\alpha_i \sin\left(\sqrt{1-\zeta_i^2}\,\Omega_i\tau'\right) + \beta_i\cos\left(\sqrt{1-\zeta_i^2}\,\Omega_i\tau'\right)\right]e^{-\zeta_i\Omega_i\tau'} \\
&\quad + \frac{h_i}{p_i}R\cos[\Omega(\tau'+\tau_0) - \theta_i], \\
\alpha_i &= \frac{1}{\sqrt{1-\zeta_i^2}\,\Omega_i}\left[\zeta_i\Omega_i\beta_i + \frac{h_i}{p_i}R\Omega\sin(\Omega\tau_0 - \theta_i) + A_0\Omega\delta_{i1}\right], \\
\beta_i &= -\frac{h_i}{p_i}R\cos(\Omega\tau_0 - \theta_i), \\
p_i &= [(\Omega_i^2 - \Omega^2)^2 + 4\zeta_i^2\Omega_i^2\Omega^2]^{1/2}, \\
\theta_i &= \tan^{-1}\frac{2\zeta_i\Omega_i\Omega}{\Omega_i^2 - \Omega^2}, \\
&\text{and} \\
h_i &= \frac{1}{M_i}\int_0^1 f(\xi)\varphi_i(\xi)\,d\xi.
\end{aligned}\right\} \tag{66}
$$

2. $\underline{\tau_s \le \tau \le \tau_e}$ (In contact with stop)

The displacement and velocity of the first regime at $\tau = \tau_s$ will be taken as the initial conditions for the second regime. At $\tau = \tau_s$, i.e., $\tau' = \tau_s - \tau_0$,

$$\left.\begin{aligned} w(\xi,\tau')\big|_{\tau'=\tau_s-\tau_0} &= \sum_i q_i(\tau_s-\tau_0)\varphi_i(\xi) \\ &\text{and} \\ \dot{w}(\xi,\tau')\big|_{\tau'=\tau_s-\tau_0} &= \sum_i \dot{q}_i(\tau_s-\tau_0)\varphi_i(\xi) \end{aligned}\right\}. \tag{67}$$

In this regime, the motion is described by Eqs. 20-26. If we let $\tau'' = \tau - \tau_s$, i.e.,

$$\tau'' = \tau' - (\tau_s - \tau_0), \tag{68}$$

the initial conditions for $\overline{w}(\xi,\tau'')$ can be written

$$\left.\begin{aligned} \overline{w}(\xi,0) &= 0 \\ &\text{and} \\ \dot{\overline{w}}(\xi,0) &= \sum_i \dot{q}_i(\tau_s-\tau_0)\varphi_i(\xi) \end{aligned}\right\}, \tag{69}$$

and the forcing function becomes

$$\overline{Q}(\xi,\tau'') = Rf(\xi)\cos[\Omega(\tau''+\tau_s)] - \sum_i \Omega_i^2 q_i(\tau_s-\tau_0)\varphi_i(\xi). \tag{70}$$

If we use Eqs. 69 and 70 and follow the analysis given in Section II, the solution in the second regime is

$$w(\xi,\tau'') = \sum_i \overline{q}_i(\tau'')\overline{\varphi}_i(\xi) + \sum_i q_i(\tau_s-\tau_0)\varphi_i(\xi), \tag{71}$$

where

$$\begin{aligned} \overline{q}_i(\tau'') &= \left[\overline{\alpha}_i \sin\left(\sqrt{1-\overline{\zeta}_i^2}\,\overline{\Omega}_i\tau''\right) + \overline{\beta}_i\cos\left(\sqrt{1-\overline{\zeta}_i^2}\,\overline{\Omega}_i\tau''\right)\right]e^{-\overline{\zeta}_i\overline{\Omega}_i\tau''} \\ &\quad + \frac{\overline{h}_i}{\overline{p}_i}R\cos[\Omega(\tau'+\tau_s) - \overline{\theta}_i], \\ \overline{\alpha}_i &= \frac{1}{\sqrt{1-\overline{\zeta}_i^2}\,\overline{\Omega}_i}\left[g_i + \frac{\overline{h}_i}{\overline{p}_i}R\Omega\sin(\Omega\tau_s - \overline{\theta}_i) + \overline{\zeta}_i\overline{\Omega}_i\overline{\beta}_i\right], \end{aligned} \tag{72}$$

(Contd.)

$$\left.\begin{aligned}
\bar{\beta}_i &= \frac{d_i}{\bar{\Omega}_i^2} - \frac{\bar{h}_i}{\bar{p}_i} R \cos\,(\Omega\tau_s - \bar{\theta}_i), \\
\bar{p}_i &= \left[\left(\bar{\Omega}_i^2 - \Omega^2\right)^2 + 4\bar{\zeta}_i^2\bar{\Omega}_i^2\Omega^2\right]^{1/2}, \\
\bar{\theta}_i &= \tan^{-1}\frac{2\bar{\zeta}_i\bar{\Omega}_i\Omega}{\bar{\Omega}_i^2 - \Omega^2}, \\
\bar{h}_i &= \frac{1}{\bar{M}_i}\int f(\xi)\bar{\varphi}_i(\xi)\,d\xi, \\
g_i &= \frac{1}{\bar{M}_i}\sum_n \dot{q}_n(\tau_s - \tau_0)C_{ni}, \\
d_i &= \frac{1}{\bar{M}_i}\sum_n C_{ni}q_n(\tau_s - \tau_0)\Omega_n^2, \\
&\text{and} \\
C_{ni} &= \int_0^1 \varphi_n(\xi)\bar{\varphi}_i(\xi)\,d\xi.
\end{aligned}\right\} \qquad \text{(Contd.) (72)}$$

The motion of the beam is completely described in Eqs. 65 and 71. If we use the stress-displacement relation from the theory of elasticity, the dynamic bending stress σ and shearing force V are given by

$$\left.\begin{aligned}
\sigma(\xi,\tau) &= C_\sigma(\xi,\tau)\frac{Ez}{\ell}\epsilon \\
&\text{and} \\
V(\xi,\tau) &= C_V(\xi,\tau)\frac{EI}{\ell^2}\epsilon
\end{aligned}\right\}, \qquad (73)$$

where

for $\tau_0 \le \tau \le \tau_s$, or $0 \le \tau' \le \tau_s - \tau_0$,

$$\left.\begin{aligned}
C_\sigma &= \frac{1}{\epsilon}\sum_i q_i(\tau')\varphi_i''(\xi) \\
&\text{and} \\
C_V &= \frac{1}{\epsilon}\sum q_i(\tau')\varphi_i'''(\xi)
\end{aligned}\right\}; \qquad (74)$$

for $\tau_s \le \tau \le \tau_e$, or $0 \le \tau'' \le \tau_e - \tau_s$,

$$\left.\begin{aligned} C_\sigma &= \frac{1}{\epsilon}\left[\sum_i \bar{q}_i(\tau'')\bar{\varphi}_i''(\xi) + \sum_i q_i(\tau_s - \tau_0)\varphi_i''(\xi)\right] \\ &\text{and} \\ C_V &= \frac{1}{\epsilon}\left[\sum_i \bar{q}_i(\tau'')\bar{\varphi}_i'''(\xi) + \sum_i q_i(\tau_s - \tau_0)\varphi_i'''(\xi)\right] \end{aligned}\right\}; \tag{75}$$

and z is the distance between the neutral axis and the point considered. The impact at the stop is

$$\left.\begin{aligned} I_0(\tau') &= 0 \text{ for } 0 \le \tau' \le \tau_s - \tau_0 \\ &\text{and} \\ I_0(\tau'') &= V(\xi,\tau'')\Big|_{\xi=r^+} + V(\xi,\tau'')\Big|_{\xi=r^-} \text{ for } 0 \le \tau'' \le \tau_e - \tau_s \end{aligned}\right\}. \tag{76}$$

For a midspan stop, Eq. 76 reduces to

$$I_0 = C_I \frac{EI}{\ell^2}\epsilon, \tag{77}$$

where

$$\left.\begin{aligned} C_I &= 0, && 0 \le \tau' < \tau_s - \tau_0 \\ &= 2C_V(\tfrac{1}{2},\tau''), && 0 \le \tau'' < \tau_e - \tau_s. \end{aligned}\right\} \tag{78}$$

V. NUMERICAL RESULTS

The theory is presented for a simply-supported beam. If the natural frequencies, normal modes, and other related parameters of the simply-supported case are replaced by those corresponding to other end conditions (see Appendix A), the results are applicable. Numerical results are given for both the simply-supported and the built-in beams.

The steady-state responses to a uniformly distributed load [$f(\xi) = 1$] have been computed using Eqs. 51 and are given in Figs. 4-7. When the amplitude R/ϵ of the forcing function is small, the response is in the range $-1 \le q/\epsilon \le 1$; this is the same as that of a vibrating beam without stops. When R/ϵ is very large, the response is like a two-span continuous beam.

For the intermediate value of amplitude R/ε, the response possesses the jump phenomenon. This occurs when the driving frequency is in the range $\Omega_1 \leq \Omega \leq \bar{\Omega}_1$ and is illustrated in Figs. 6 and 7, where the amplitude follows the curve ABC, then jumps down to E, and then on to F as the forcing frequency is increased; conversely, when the forcing frequency is slowly decreased, the jump occurs at D. The nondimensional amplitude, $w(\xi, \tau)$, is computed by substituting the normal coordinate, q, obtained from Fig. 4 or 5, into Eq. 36 and using the appropriate normal mode as given in Appendix A.

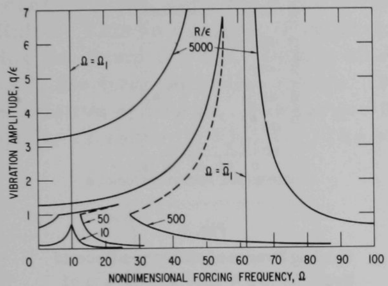

Fig. 4. Amplitude-response Curves of a Simply-supported Beam for a Damping Ratio of 0.1 ($\varepsilon = e/\ell$; see Eqs. 36 for the displacement of the beam). ANL Neg. No. 113-3280.

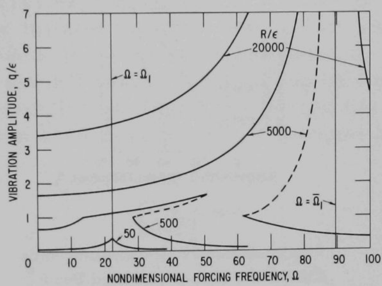

Fig. 5. Amplitude-response Curves of a Built-in Beam for a Damping Ratio of 0.1 ($\varepsilon = 0.81\ e/\ell$; see Eqs. 36 for the displacement of the beam). ANL Neg. No. 113-3275.

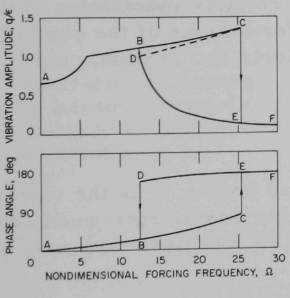

Fig. 6. Phase-response Curve of a Simply-supported Beam for $R/\varepsilon = 50$ and a Damping Ratio of 0.1 ($\varepsilon = e/\ell$). ANL Neg. No. 113-3289.

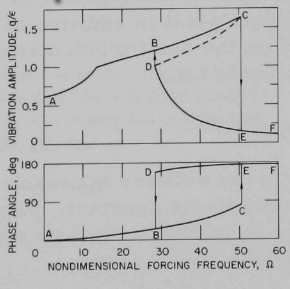

Fig. 7. Phase-response Curve of a Built-in Beam for $R/\varepsilon = 500$ and a Damping Ratio of 0.1 ($\varepsilon = 0.81\ e/\ell$). ANL Neg. No. 113-3285.

The dynamic stresses are given in Eqs. 73. The shearing force at $\xi = \frac{1}{2}$, bending stresses at $\xi = \frac{1}{4}$ and $\frac{1}{2}$ are shown in Figs. 8 and 9 when $\tau = \tau_e$, i.e., at the extreme position. Those are the maximum stresses; they are quite sensitive to the driving frequency.

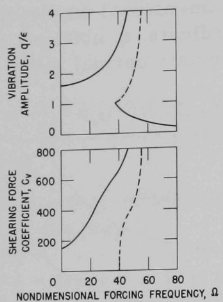

Fig. 8

Amplitude Response and Shearing Force Curves of a Simply-supported Beam for $R/\varepsilon = 1000$ and $\zeta_1 = 0.01$ ($\varepsilon = e/\ell$). ANL Neg. No. 113-3279.

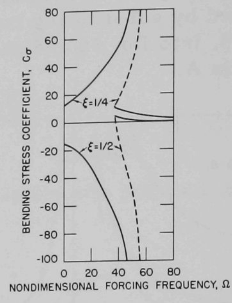

Fig. 9

Bending Stresses of a Simply-supported Beam for $R/\varepsilon = 1000$ and $\zeta_1 = 0.01$ ($\varepsilon = e/\ell$). ANL Neg. No. 113-3276.

VI. CHARACTERISTICS OF THE IMPACT

As indicated earlier, the analysis is based on the assumption that there is no relative motion between the beam and the stop at the contact point. In reality, the two bodies suffer a relative indentation in the vicinity of the impact point in addition to bulk deformations of the objects as a whole. For the elastic indentation, the Hertz law is widely used in the dynamic case, i.e.,

$$\alpha = KP^{2/3}, \tag{79}$$

where α is the relative approach of the two bodies, P is the contact force, and K is the Hertz constant. For particular geometries, such as two cylinders in contact with their axes parallel, the Hertz law may be approximated by the linear form

$$\alpha = K_s P. \tag{80}$$

To investigate the effect of the local deformation at the contact area, the impact developed by the instantaneous arrest of a moving beam was analyzed (see Appendix B). A beam having an initial velocity u is instantaneously brought to rest at its two ends. At the midpoint, it strikes a spring with stiffness K_s. The impact and the bending stress are functions of the relative stiffness K_s/K_b, where K_b is the stiffness of the beam. From Eqs. B.13,

$$\left.\begin{aligned} \sigma(\xi,\tau) &= C_\sigma\left(\frac{Em}{I}\right)^{1/2} zu \\ \text{and} \\ I_0(\tau) &= C_I(EIm)^{1/2}\frac{u}{\ell} \end{aligned}\right\}. \tag{81}$$

Plots of the coefficients C_I and C_σ at $\xi = \frac{1}{2}$ are shown in Figs. 10 and 11. There are double peaks if K_s/K_b is large. If the two bodies are perfectly rigid at contact point, i.e., $K_s/K_b = \infty$, the first peak occurs at $\tau = 0$ and is infinite. This is because it requires an infinite force of zero duration to bring the beam to rest. On the other hand, if the relative stiffness is small, the first peak disappears. The second peak of C_I also depends on the relative stiffness. The larger the relative stiffness, the larger the impact, as shown in Fig. 12. The second peak occurs when τ is equal to

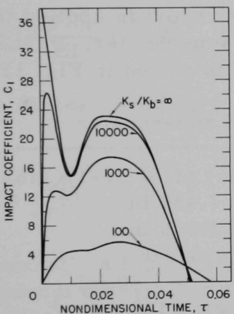

Fig. 10

History of Impact Developed by Instantaneous Arrest of a Moving Beam. ANL Neg. No. 113-3273 Rev. 1.

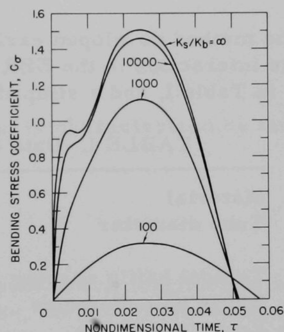

Fig. 11

History of Bending Stress at Midspan Developed by the Instantaneous Arrest of a Moving Beam. ANL Neg. No. 113-3287.

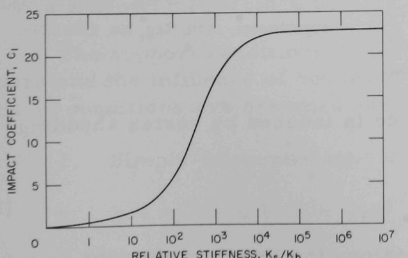

Fig. 12

Impact on a Spring at Midpoint Developed by the Instantaneous Arrest of a Moving Beam. ANL Neg. No. 113-3283.

a quarter of the period of fundamental mode of the beam. From Fig. 11, the bending stress possesses only one peak, except when the stop is rigid; however, the first peak is smaller than the second. The difference in the histories between the bending stress and the impact comes from the fact that it requires a finite time, dependent on the wave velocity, to reach the maximum response for the bending stress. But if the stop is quite rigid, it develops a large impact instantaneously and the force is "localized" in the impact area.

Figure 12 indicates that if K_s/K_b is larger than 10^4, a rigid stop can be assumed. In such a circumstance, the energy required to produce the local deformation is only a small fraction of the total energy, and the assumption of negligible local deformation is justified.

VII. APPLICATION: TUBE-BAFFLE IMPACT IN EBR-II SUPERHEATER

The method developed earlier in this report is applied to study tube-baffle interaction in the EBR-II steam superheater. Pertinent data[11] are given in Table I, and a simplified model is shown in Fig. 13.

TABLE I. Data for EBR-II Superheater

Material	Croloy ($2\frac{1}{4}$% Cr-1% Mo)
Tube diameter	d_o = 1.438 in.
	d_i = 1.065 in.
Typical baffle spacing	$\ell/2$ = 6.5 ft
Baffle hole diameter	$d_o + 2e$ = 1.50 in.
Young's modulus	$E = 26 \times 10^6$ lb/in.^{2}
Material density	0.28 lb/in.^{3}
Mean crossflow velocity	U_0 = 5.26 ft/sec

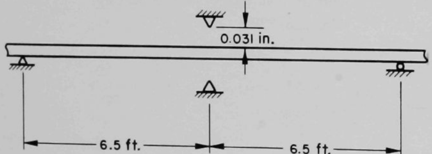

Fig. 13

Simplified Model of Tube Baffle in EBR-II Superheater. ANL Neg. No. 113-3290.

The distributed transverse force is induced by vortex shedding and can be approximated by

$$F(x,t) = \tfrac{1}{2}C_k\rho U^2 d_o \sin \omega t. \tag{82}$$

The distribution of crossflow velocity along the length of the tube is not expected to be uniform. To determine the effects of nonuniform distribution, the velocity is expressed in the form

$$U^2(x) = U_0^2 f(x). \tag{83}$$

The distributed force can now be written

$$F(x,t) = (\tfrac{1}{2}C_k \rho U_0^2 d_o) f(x) \sin \omega t \tag{84}$$

and, in dimensionless form, as

$$Q(\xi, \tau) = R f(\xi) \sin \Omega\tau, \tag{85}$$

where

$$\left.\begin{aligned} R &= \frac{1}{2}\frac{C_k \rho U_0^2 d_o \ell^3}{EI}, \\ &\text{and} \\ \Omega &= \left(\frac{M+m}{EI}\right)^{1/2} \omega \ell^2. \end{aligned}\right\} \tag{86}$$

The vortex-shedding frequency, ω, is characterized by the Strouhal number

$$N_s = \frac{\omega d_o}{2\pi U_0} \tag{87}$$

and is additionally a function of the transverse and longitudinal spacing between adjacent tubes. For a single tube, the vortex-shedding frequency is characterized by a Strouhal number of 0.2. For the tube-spacing and flow orientation in the EBR-II steam superheater, the vortex-shedding frequency (based on an experimental study by Chen[12]) is represented by a Strouhal number of 0.66.[11] The dimensionless forcing frequencies, Ω, corresponding to these Strouhal numbers are 16 ($N_s = 0.2$; single-tube value) and 52.8 ($N_s = 0.66$; tube-bank value), as computed from Eq. 87. If we assume a lift coefficient of unity ($C_k = 1$) and use the data of Table I, Eqs. 86 yield $R = 0.230$.

The support conditions at the two ends are not known. To better understand the influence of boundary conditions, both simply-supported and built-in conditions are assumed and the following cases are studied:

1. Simply-supported tube with uniform flow distribution [$f(x) = 1$];

   $\epsilon = e/\ell$ and $R/\epsilon = 1157$.

2. Simply-supported tube with nonuniform flow distribution, assuming

$$f(x) = \frac{\pi}{2} \sin \frac{\pi x}{\ell}; \quad \varepsilon = e/\ell \text{ and } R/\varepsilon = 1157.$$

3. Built-in tube with uniform flow distribution $[f(x) = 1]$;

$$\varepsilon = 0.81(e/\ell) \text{ and } R/\varepsilon = 1428.$$

The amplitude-response curves are shown in Fig. 14, where a damping ratio of 0.01 is used. The corresponding deflection, shearing force, and bending stress of the tube when it is at the two extreme baffle locations are given in Figs. 15-17.

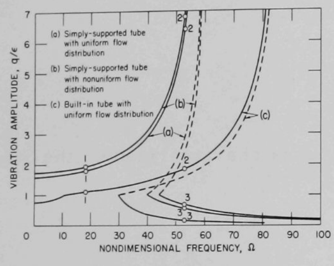

Fig. 14. Amplitude-response Curves of Tubes in EBR-II Superheater. ANL Neg. No. 113-3272 Rev. 1.

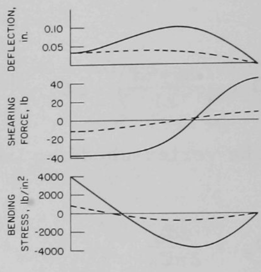

Fig. 15. Dynamic Responses of a Simply-supported Tube in EBR-II Superheater Excited by Uniform Crossflow. Solid lines for Ω = 52.8; dotted lines for Ω = 16.0. ANL Neg. No. 113-3274.

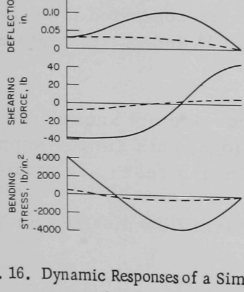

Fig. 16. Dynamic Responses of a Simply-supported Tube in EBR-II Superheater Excited by Nonuniform Crossflow. Solid lines for Ω = 52.8; dotted lines for Ω = 16.0. ANL Neg. No. 113-3284.

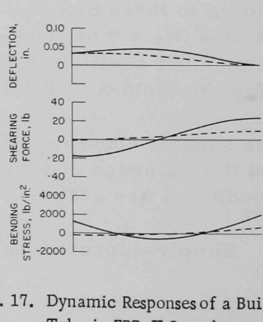

Fig. 17. Dynamic Responses of a Built-in Tube in EBR-II Superheater Excited by Uniform Crossflow. Solid lines for Ω = 52.8; dotted lines for Ω = 16.0 ANL Neg. No. 113-3277.

For a single tube ($\Omega = 16$), the stress level is very low; this is because the forcing frequency is far from the resonant frequency, as shown in Fig. 14 as operating point 1. However, for the tube bank ($\Omega = 52.8$), the stress is relatively high for the simply-supported tube; this is because the forcing frequency is almost the same as the resonant frequency. If the tube is fixed at its two ends, the maximum stress is reduced by about 50%. This reduction is attributed to the increase in natural frequency of the tube.

A few remarks should be made concerning the responses in this example: (a) In Figs. 15-17, the curves plotted with solid lines correspond to point 2 in Fig. 14. At this driving frequency, there is another stable point (point 3) at which the stress is very small. Whether the tube responds at point 2 or 3 depends on the increase or decrease of the driving frequency, as discussed in Section V; the response at point 2 represents the worst case. (b) The responses for uniform and nonuniform flow distribution are quite similar. (c) Comparing the responses between simply-supported and built-in ends illustrates that boundary conditions have a significant influence. (d) The examples given here demonstrate the method of analysis. The result is based on the assumption that the lift coefficient C_k is well understood, and its value has been taken as unity. However, at this time the essential question remaining is "What is the magnitude of fluid dynamic force when the fluid is flowing past a vibrating tube?" This must be answered before the problem of crossflow-induced vibration can be solved completely.

VIII. CONCLUDING REMARKS

1. A method has been presented for studying the dynamic response of a beam with deflection resistors. This method can be applied to beams with various boundary conditions. By the same approach, it is easily extended to other elastic systems such as plate-vibration problems.

2. The stops play an important role for certain ranges of R/ϵ in which the system possesses the properties associated with nonlinear vibration. For a small value of R/ϵ, the beam can be treated as one without stops, and its resonant frequency is Ω_1. For a large value of R/ϵ, the system becomes a two-span continuous beam, and the resonant frequency is $\overline{\Omega}_1$. Within the intermediate range of R/ϵ, the resonant frequency is amplitude-dependent and is between Ω_1 and $\overline{\Omega}_1$.

3. The most unfavorable condition is the coincidence of the forcing frequency and the natural frequency of the system. In any case, periodic disturbances near the natural frequency of the system should be suppressed in order to reduce the amplitude of vibration, e.g., by reducing the span, increasing the rigidity, or changing the support condition.

4. The assumption of neglecting the local deformation at the stops is justified, provided the stops are relatively strong; more precisely, $K_s/K_b > 10^4$. In these cases, the energy associated with the local deformation is only a small fraction of the total energy.

5. To understand more thoroughly the response due to tube-baffle interaction in the EBR-II superheater, the intrinsic problem to be solved is the lifting coefficient of crossflow, especially (1) the fundamental vortex-shedding phenomenon associated with vibrating tubes in a tube bank and (2) fluid-structure coupling.

6. If either the geometry or the forcing function is not symmetric, more modes should be included in finding the "gross" response.

APPENDIX A

Eigenfunctions, Natural Frequencies, and Related Beam-vibration Data

1. Simply-supported Beam (see Fig. A.1)

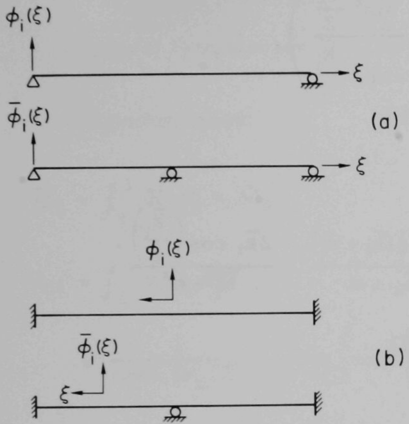

Fig. A.1

Coordinates for (a) Simply-supported and (b) Built-in Beams. ANL Neg. No. 113-3281.

a. No Contact with Stop

Characteristic equation: $\sin k_i = 0$

i	1	2	3	4	5
k_i	π	2π	3π	4π	5π

Natural frequencies: $\Omega_i = k_i^2, \quad i = 1, 2, 3, \ldots$

Eigenfunctions: $\varphi_i(\xi) = \sin(i\pi\xi)$

$$\varphi_1(\tfrac{1}{2}) = 1$$

$$M_i = \int_0^1 \varphi_i^2 \, d\xi = \frac{1}{2}$$

b. Having Contact with Stop

Characteristic equation: $\tan\left(\frac{\bar{k}_i}{2}\right) = \tanh\left(\frac{\bar{k}_i}{2}\right)$

i	1	2	3	4	5
k_i	7.86	14.1	20.4	26.7	33.0

Natural frequencies: $\bar{\Omega}_i = \bar{k}_i^2, \quad i = 1, 2, 3, \ldots$

Eigenfunctions: $\bar{\varphi}_i(\xi) = \sin(\bar{k}_i \xi) - \dfrac{\sin(\bar{k}_i/2)}{\sinh(\bar{k}_i/2)} \sinh(\bar{k}_i \xi)$

$$\bar{M}_i = \int_0^1 \bar{\varphi}_i^2(\xi)\, d\xi = \frac{1}{2}\left(1 - \frac{\sin^2 \frac{\bar{k}_i}{2}}{\sinh^2 \frac{\bar{k}_i}{2}}\right)$$

$$C_{ni} = \int_0^1 \varphi_n \bar{\varphi}_i \, d\xi$$

$$= \frac{\sin[\frac{1}{2}(\bar{k}_i - \pi)]}{\bar{k}_i - \pi} - \frac{\sin[\frac{1}{2}(\bar{k}_i + \pi)]}{\bar{k}_i + \pi} - \frac{2\bar{k}_i \cos\left(\frac{\bar{k}_i}{2}\right)}{\bar{k}_i^2 + \pi^2}$$

2. Fixed-Fixed Beam (see Fig. A.1)

a. No Contact with Stop

Characteristic equation: $\cos(k_i)\cosh(k_i) = 1$

i	1	2	3	4	5
k_i	4.73	7.85	11.0	14.1	17.3

Natural frequencies: $\Omega_i = k_i^2$

Eigenfunctions: $\varphi_i(\xi) = \dfrac{\cosh(k_i \xi)}{\cosh(k_i/2)} - \dfrac{\cos(k_i \xi)}{\cos(k_i/2)}$

$$\varphi_1(0) = 1.239$$

$$M_i = \int_0^1 \varphi_i^2(\xi)\, d\xi = \frac{1}{2\cosh^2(k_i/2)}\left[\frac{1}{k_i}\sinh(k_i) + 1\right]$$

$$+ \frac{1}{2\cos^2(k_i/2)}\left[\frac{1}{k_i}\sin(k_i) + 1\right]$$

$$- \frac{2}{k_i}\left[\tanh(k_i/2) + \tan(k_i/2)\right]$$

b. <u>Having Contact with Stop</u>

Characteristic equation: $\cos(\bar{k}_i/2)\cosh(\bar{k}_i/2) = 1$

i	1	2	3	4	5
k_i	9.46	15.7	22.0	28.2	34.6

Natural frequencies: $\bar{\Omega}_i = \bar{k}_i^2$

Eigenfunctions: $\bar{\varphi}_i(\xi) = \dfrac{\cosh(\bar{k}_i\xi)}{\cosh(\bar{k}_i/4)} - \dfrac{\cos(\bar{k}_i\xi)}{\cos(\bar{k}_i/4)}$

$$\overline{M}_i = \int_0^1 \bar{\varphi}_i^2 \, d\xi = M_i$$

$$C_{ni} = \int_0^1 \varphi_n \bar{\varphi}_i \, d\xi$$

$$= \frac{1}{\cosh(k_n/2)} \frac{1}{k_n + \bar{k}_i} \left(\sinh\left[\frac{1}{2}(k_n + \bar{k}_i)\right] \right.$$

$$\left. - \tanh(\bar{k}_i/4)\left\{\cosh\left[\frac{1}{2}(k_n + \bar{k}_i)\right] - 1\right\}\right)$$

$$+ \frac{1}{\cos(k_n/2)} \frac{1}{k_n + \bar{k}_i} \left(\sin\left[\frac{1}{2}(k_n + \bar{k}_i)\right] \right.$$

$$\left. - \tan(\bar{k}_i/4)\left\{\cos\left[\frac{1}{2}(k_n + \bar{k}_i)\right] - 1\right\}\right)$$

$$- 2\tan(\bar{k}_i/4)\frac{1}{k_n^2 + \bar{k}_i^2}\left\{k_n \tanh(k_n/2)\sin(\bar{k}_i/2)\right.$$

$$\left. - \bar{k}_i \cos(\bar{k}_i/2) + \frac{\bar{k}_i}{\cosh(k_n/2)}\right\}$$

$$- 2\tanh(\bar{k}_i/4)\frac{1}{k_n^2 + \bar{k}_i^2}\left[-k_n \tan(k_n/2)\sinh(\bar{k}_i/2)\right.$$

$$\left. - \bar{k}_i \cosh(\bar{k}_i/2) + \frac{\bar{k}_i}{\cos(k_n/2)}\right]$$

$$- \frac{2\bar{k}_i}{k_n^2 + \bar{k}_i^2}\left[\sinh(\bar{k}_i/2) - \tanh(\bar{k}_i/4)\cosh(\bar{k}_i/2)\right.$$

$$+ \frac{\tanh(\bar{k}_i/4)}{\cos(k_n/2)} + \sin(\bar{k}_i/2)\Big]$$

$$+ \frac{2k_n}{k_n^2 + \bar{k}_i^2} \, [\tanh(\bar{k}_i/4) \tan(k_n/2) \sinh(\bar{k}_i/2)$$

$$+ \tan(k_n/2) \cosh(\bar{k}_i/2) - \tanh(k_n/2) \cos(\bar{k}_i/2)]$$

$$+ c'_{ni}$$

where

$$c'_{ni} = \frac{1}{2}\left[\frac{1}{\cosh(k_n/2)} + \frac{1}{\cos(k_n/2)}\right], \qquad \text{if } k_n = \bar{k}_i,$$

and

$$c'_{ni} = \frac{1}{\cosh(k_n/2)} \frac{1}{\bar{k}_i - k_n} \left(\sinh\left[\frac{1}{2}(\bar{k}_i - k_n)\right]\right.$$

$$\left. - \tanh(\bar{k}_i/4)\left\{\cosh\left[\frac{1}{2}(\bar{k}_i - k_n)\right] - 1\right\}\right)$$

$$+ \frac{1}{\cos(k_n/2)} \frac{1}{\bar{k}_i - k_n} \left(\sin\left[\frac{1}{2}(\bar{k}_i - k_n)\right]\right.$$

$$\left. - \tan(\bar{k}_i/4)\left\{\cos\left[\frac{1}{2}(\bar{k}_i - k_n)\right] - 1\right\}\right), \qquad \text{if } k_n \neq \bar{k}_i.$$

APPENDIX B

Dynamic Stresses and Impact Developed by the Instantaneous Arrest of a Moving Beam

When a beam under uniform translation at a velocity u is brought to rest instantaneously at two ends and hits a spring stop at its midpoint (see Fig. B.1), an infinite number of modes of vibration are excited. This problem is analyzed to investigate the stresses produced in the beam and the impact at the stop. The field equations are

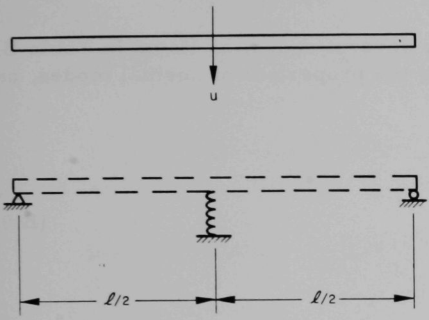

Fig. B.1. A Moving Beam Striking a Spring Stop at Midspan. ANL Neg. No. 113-3288.

$$\frac{\partial^4 w}{\partial \xi^4} + \alpha \frac{\partial^5 w}{\partial \xi^4 \partial \tau} + \beta \frac{\partial w}{\partial \tau} + \frac{\partial^2 w}{\partial \tau^2} = 0, \tag{B.1}$$

$$\left.\begin{aligned} w(0,\tau) &= w''(0,\tau) = 0, \\ w(1,\tau) &= w''(1,\tau) = 0, \end{aligned}\right. \tag{B.2}$$

$$w'''(\tfrac{1}{2},\tau) = 24R_s w(\tfrac{1}{2},\tau), \tag{B.3}$$

$$w(0,\tau) = 0,$$

and

$$\dot{w}(0,\tau) = v, \tag{B.4}$$

where

$$\left.\begin{aligned} v &= \left(\frac{m}{EI}\right)^{1/2} u\ell, \\ R_s &= K_b/K_s, \end{aligned}\right\} \tag{B.5}$$

and K_b is the stiffness constant of the beam ($K_b = 48\,EI/\ell^3$). With a modal method of analysis, the solution is written

$$w(\xi,\tau) = \sum_i q_i(\tau)\varphi_i(\xi). \tag{B.6}$$

Since u is uniform across the length of the beam, only symmetric modes are excited. The field equations indicate that the mode-shape functions are

$$\varphi_i(\xi) = \sin(k_i \xi) - \frac{\cos(k_i/2)}{\cosh(k_i/2)} \sinh(k_i \xi), \tag{B.7}$$

where k_i is the solution of the equation

$$\tan(k_i/2) = \tanh(k_i/2) - \frac{k_i^3}{12R_s}. \tag{B.8}$$

The value of k_i has been computed for several cases, as listed in Table B.I. Substituting Eq. B.6 into B.1 and using the properties of normal modes, one finds for the normal coordinates

$$\left.\begin{aligned} \ddot{q}_i + 2\zeta_i\dot{q}_i + \Omega_i^2 q_i &= 0, \\ q_i(0) &= 0, \\ \text{and} \qquad \dot{q}_i(0) &= \frac{S_i}{M_i} v, \end{aligned}\right\} \tag{B.9}$$

where

$$\left.\begin{aligned} M_i &= \int_0^1 \varphi_i^2(\xi)\, d\xi, \\ S_i &= \int_0^1 \varphi_i(\xi)\, d\xi, \\ \zeta_i &= \frac{\alpha\Omega_i}{2} + \frac{\beta}{2\Omega_i}, \\ \text{and} \qquad \Omega_i &= k_i^2. \end{aligned}\right\} \tag{B.10}$$

TABLE B.I. Eigenvalues for a Simply-supported Beam with an Elastic Support at Midpoint

	R_s					
k_i	1	10	100	1,000	10,000	100,000
k_1	1.8612	2.7568	3.7215	3.9062	3.924	3.9263
k_2	4.7267	4.8614	5.9182	6.9436	7.0567	7.0673
k_3	7.8570	7.8855	8.2099	9.7840	10.173	10.206
k_4	10.996	11.006	11.117	12.323	13.267	13.343
k_5	14.137	14.142	14.192	14.840	16.325	16.478
k_6	17.279	17.281	17.308	17.638	19.327	19.609
k_7	20.420	20.422	20.438	20.623	22.250	22.735
k_8	23.562	23.563	23.573	23.688	25.080	25.857
k_9	26.703	26.704	26.711	26.787	27.858	28.971
k_{10}	29.845	29.845	29.850	29.904	30.670	32.078

The solution to Eqs. B.9 takes one of three forms, depending on the magnitude of ζ_i

$$q_i(\tau) = \frac{S_i}{M_i} v\bar{q}_i(\tau), \tag{B.11}$$

where

$$\left.\begin{aligned} &\text{for } \zeta_i > 1, \ \bar{q}_i(\tau) = \frac{1}{\sqrt{\zeta_i^2 - 1}\,\Omega_i} e^{-\zeta_i\Omega_i\tau} \sinh\left(\sqrt{\zeta_i^2 - 1}\,\Omega_i\tau\right), \\ &\text{for } \zeta_i = 1, \ \bar{q}_i(\tau) = \tau e^{-\Omega_i\tau}, \\ \text{or} \quad &\text{for } \zeta_i < 1, \ \bar{q}_i(\tau) = \frac{1}{\sqrt{1 - \zeta_i^2}\,\Omega_i} e^{-\zeta_i\Omega_i\tau} \sin\left(\sqrt{1 - \zeta_i^2}\,\Omega_i\tau\right). \end{aligned}\right\} \tag{B.12}$$

The bending stress of the beam and the impact at spring are

$$\left.\begin{aligned} \sigma(\xi, \tau) &= C_\sigma\left(\frac{Em}{I}\right)^{1/2} zu \\ \text{and} \quad I_0(\tau) &= C_I(EIm)^{1/2}\frac{u}{\ell} \end{aligned}\right\}, \tag{B.13}$$

where

$$\left.\begin{aligned} C_\sigma &= \sum_i \bar{q}_i(\tau)\varphi_i''(\xi)\frac{S_i}{M_i} \\ \text{and} \quad C_I &= 2\sum_i \bar{q}_i(\tau)\varphi_i'''\left(\frac{1}{2}\right)\frac{S_i}{M_i} \end{aligned}\right\}. \tag{B.14}$$

The bending stress at a given percentage of distance from the stop is independent of the length of the beam; the impact varies inversely with the length of the beam. This is because when the beam is shorter, the natural frequencies will be higher and the time required to bring the beam to stop will be shorter, thus requiring a larger force at the stop.

REFERENCES

1. S. Timoshenko and D. H. Young, *Vibration Problems in Engineering*, D. Van Nostrand Co. Inc., New York, 3rd Ed., p. 411 (1955).

2. A. Cemal Eringen, *Transverse Impact on Beams and Plates*, J. Appl. Mech. *20*, p. 461 (1953).

3. K. E. Barnhart, Jr., and W. Goldsmith, *Stresses in Beams during Transverse Impact*, J. Appl. Mech. *24*, p. 440 (1957).

4. Werner Goldsmith, *Impact*, Edward Arnold Ltd., London (1960).

5. Raymond L. Bisplinghoff, Holt Ashley, and Robert L. Halfman, *Aeroelasticity*, Addison-Wesley Publishing Co. Inc., Reading, Mass. (1957).

6. T. H. H. Pian and H. I. Flomenhoff, *Analytical and Experimental Studies on Dynamic Loads in Airplane Structures during Landing*, J. Aeron. Sci. *13*, No. 12 (Dec 1950).

7. K. W. Deas, *Dynamic Stresses in Free-pinned Beams during Impact with a Stop*, J. Mech. Eng. Sci. *10*, p. 282 (1968).

8. A. E. H. Love, *Mathematical Theory of Elasticity*, 4th Ed., Cambridge University Press, p. 198 (1927).

9. A.V. Srinivasan, *Steady-state Response of Beams Supported on Nonlinear Spring*, J. AIAA *4*, p. 1863 (1966).

10. Austin Blaquiére, *Nonlinear System Analysis*, Academic Press, New York and London, p. 90 (1966).

11. M. W. Wambsganss, Jr., *Evaluation of Potential Tube Vibration in EBR-II Steam Superheaters and Evaporators at Full Power*, ANL-7600 (Nov 1969).

12. Y. N. Chen, *Flow-Induced Vibration and Noise in Tube-bank Heat-exchangers due to von Karman Streets*, J. Eng. Ind., Trans. ASME, Series B *90*(1), 134-146 (Feb 1968).

X